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Affine action and Margulis invariant

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Abstract

In this paper, we show that two Zariski dense subgroups consisting of hyperbolic elements in $SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$ with the same marked Margulis invariant, are conjugate. We also consider in affine deformations an analogue of quasifuchsian deformation of Fuchsian groups.
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1. Introduction

Let M be a complete flat affine manifold. Milnor [15] conjectured that $\pi_1(M)$ is virtually polycyclic. But Margulis found counterexamples with free fundamental groups in [11,12,1,2] using *Margulis invariant*. In this notes we want to closely investigate this invariant. Specially we concern an affine isometry action on the space $\mathbb{R}^{n+1,n}$. This space is a vector space $\mathbb{R}^{2n+1}$ with a bilinear form $\mathbb{B}$ defined by

$$\mathbb{B}(X, Y) = x_1 y_1 + \cdots + x_{n+1} y_{n+1} - x_{n+2} y_{n+2} - \cdots - x_{2n+1} y_{2n+1}.$$

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The group of isometries of this space is $O(n+1,n) \ltimes \mathbb{R}^{2n+1}$. For $(A,b) \in O(n+1,n) \ltimes \mathbb{R}^{2n+1}$, A is called a linear part, b a translational part. The nullcone

$$\mathcal{N} = \{x \in \mathbb{R}^{2n+1} \mid \mathbb{B}(x,x) = 0\}$$

is invariant by $O(n+1,n)$.

An element $g \in SO(n+1,n)$ is called (purely) *hyperbolic* if g has real eigenvalues, counting multiplicities,

$$|\lambda_{-n}(g)| \leqslant \cdots \leqslant |\lambda_{-1}(g)| < \lambda_0(g) = 1 < |\lambda_1(g)| \leqslant \cdots |\lambda_n(g)|$$

such that $\lambda_i(g)\lambda_{-i}(g) = 1$. Denote $\chi_i(g)$ the corresponding eigenvectors so that

(1) $B(\chi_i(g), \chi_j(g)) = \delta_{i,-j}$,
(2) for $i = 0$, $B(\chi_0(g), \chi_0(g)) = 1$ and $(\chi_0(g), \chi_{-n}(g), \dots, \chi_{-1}(g), \ \chi_1(g), \dots, \chi_n(g))$ is positively oriented.

Note that if we take $-\chi_i(g)$ instead of $\chi_i(g)$, then by condition 1 above, we should take $-\chi_{-i}(g)$, so the orientation remains the same.

We say (A,b) is hyperbolic if A is hyperbolic. For $h = (g,v) \in SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$ hyperbolic, the *Margulis invariant* $\alpha(h)$ of h is

$$\mathbb{B}(v, \chi_0(g)).$$

If n is odd, $\alpha(h)\alpha(h^{-1}) > 0$.

In this paper, we consider the marked Margulis invariant of a Zariski dense subgroup in $SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$. The first theorem we want to prove is

Theorem A. *Let Γ_1 and Γ_2 be Zariski dense subgroups of $SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$ consisting of hyperbolic elements. Suppose $\phi : \Gamma_1 \to \Gamma_2$ is an isomorphism preserving the Margulis invariant. Then Γ_1 and Γ_2 are conjugate. Specially if ϕ is such that $\phi(A,b) = (A,c)$, i.e., Γ_1 and Γ_2 have the identical linear parts, then ϕ is a conjugation by a translation.*

This theorem is proved in Theorems 2 and 3. When $n = 1$, the theorem is independently proved by [5]. Such type of theorem for marked length spectrum is known in [8,9,3]. But Margulis invariant comes with a sign which reflects the dynamics of an action.

An *affine deformation* of $\Gamma \subset SO(n+1,n)$ is a homomorphism $\phi : \Gamma \to SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$ such that $\phi(A,b) = (A, u_\phi(A))$. An affine deformation is *proper* if its action on $\mathbb{R}^{2n+1}$ is proper.

Up to Section 3, we deal with the case $SO(2,1) \ltimes \mathbb{R}^4$ to get a general idea for $SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$. Also the purpose of using $\mathbb{R}^4$ is to extend the action of $SO(2,1)$ on $\mathbb{R}^3$ to $\mathbb{R}^4$. Here we identify $SO(2,1)$ with

$$\begin{bmatrix} 1 & 0 \\ 0 & SO(2,1) \end{bmatrix}.$$

We want to investigate the analogue of quasifuchsian deformations of Fuchsian groups in a hyperbolic 3-space.

If $\phi : \Gamma \to SO(2,1) \ltimes \mathbb{R}^4$ is an affine deformation with $u = u_\phi \in Z^1(\Gamma, \mathbb{R}^4) \subset Z^1(\Gamma, \mathfrak{so}(3,1))$ such that

$$-\frac{tr(u(g)g)}{\sqrt{(trg-2)^2-4}} > 0$$

for all $g \in \Gamma$ hyperbolic, we call u_ϕ *positive*. For definitions, see Sections 3 and 5. Then we have

Theorem B. *Suppose $\Gamma \subset SO(2,1)$ is a cocompact lattice. Then any affine deformation $\phi : \Gamma \to SO(2,1) \ltimes \mathbb{R}^4$ with u_ϕ positive is not proper.*

2. $SO(2,1) \ltimes \mathbb{R}^4$ action on $\mathbb{R}^4$

The complement of the origin in the nullcone in $\mathbb{R}^{2,1}$ consists of two components

$$\mathcal{N}_+ = \{x \in \mathcal{N} \mid x_3 > 0\}, \quad \mathcal{N}_- = \{x \in \mathcal{N} \mid x_3 < 0\}.$$

One says that a vector $v \in \mathbb{R}^3$ is spacelike if $\mathbb{B}(v,v) > 0$, timelike if $\mathbb{B}(v,v) < 0$. We naturally embed $SO(2,1)$ into $SO(3,1)$ as

$$\begin{bmatrix} 1 & 0 \\ 0 & SO(2,1) \end{bmatrix}.$$

We can develop the same theory for any embedding of $SO(2,1)$ into $SO(3,1)$ but here we choose a standard one. A (purely) *hyperbolic* isometry A in $SO(2,1)$ has eigenvalues $\lambda^{-1}, 1, \lambda$ where $\lambda > 1$. Denote $\chi^- \in \mathcal{N}_+, \chi_0, \chi^+ \in \mathcal{N}_+$ eigenvectors of A with $\mathbb{B}(\chi_0, \chi_0) = 1$ corresponding to $\lambda^{-1}, 1, \lambda$, respectively. There exists a unique vector $\chi_0 = \chi_0(A)$ so that (χ_0, χ^-, χ^+) is a positively oriented basis of $\mathbb{R}^3$.

Let $g = (A, X) \in SO(2,1) \times \mathbb{R}^4$. Regarding A as an element of $SO(3,1)$, A will have eigenvalues $\lambda^{-1}, 1, 1, \lambda$. Denote χ_0' an eigenvector for an extra 1 perpendicular to χ_0 so that $|\chi_0'| = 1$. Actually in our case, χ_0' can be chosen as $e_1 = (1,0,0,0)$. Then $(e_1 = \chi_0', \chi_0, \chi^-, \chi^+)$ is a positively oriented basis of $\mathbb{R}^4$. We want to find an invariant axis of $g = (A, X)$ along which g translates.

Lemma 1. *Let $g = (A, X)$ as above. Set $X = k\chi_0 + le_1 + m\chi^- + n\chi^+$. Then any line of slope $\frac{l}{k}$ (with respect to (χ_0, e_1)) on the plane $x\chi_0 + ye_1 + \frac{m}{1-\lambda^{-1}}\chi^- + \frac{n}{1-\lambda}\chi^+, x, y \in \mathbb{R}$, is invariant by g.*

Proof. Let $C = x\chi_0 + ye_1 + z\chi^- + w\chi^+$ and $t(a\chi_0 + be_1) + C$ be an invariant line of g. Then

$$A(t(a\chi_0 + be_1) + C) + X = s(a\chi_0 + be_1) + C.$$

From this one obtains

$$ta + k + x = sa + x,$$

$$tb + l + y = sb + y,$$

$$z\lambda^{-1} + m = z,$$

$$w\lambda + n = w.$$

So $z = \frac{m}{1-\lambda^{-1}}$, $w = \frac{n}{1-\lambda}$. From the first two equations above, x and y are arbitrary and $s = t + \frac{k}{a} = t + \frac{l}{b}$. So $\frac{b}{a} = \frac{l}{k}$. So the claim follows. $\square$

Observe that

$$\begin{aligned} A(t(a\chi_0 + be_1) + C) + X &= \left(t + \frac{k}{a}\right)(a\chi_0 + be_1) + C \\ &= t(a\chi_0 + be_1) + (k\chi_0 + le_1) + C. \end{aligned}$$

So g translates along this line by the vector $k\chi_0 + le_1$.

Now we define the *Margulis invariant* of $g = (A, b)$ by

$$(k, l) = (\mathbb{B}(b, \chi_0), \mathbb{B}(b, e_1)) = (\chi_0 \text{ component of } b, e_1 \text{ component of } b).$$

Since $g^{-1} = (A^{-1}, -A^{-1}b)$, if $b = k\chi_0 + le_1 + m\chi^- + n\chi^+$, then

$$-A^{-1}b = -k\chi_0 - le_1 - m\lambda\chi^- - n\lambda^{-1}\chi^+.$$

Also $(-\chi_0, \chi^+, \chi^-)$ is a positively oriented basis for A^{-1}. So the Margulis invariant of g^{-1} is $(k, -l)$.

We investigate the importance of this invariant, not completely, but yet to be explored. Suppose two purely hyperbolic elements g_1 and g_2 in $SO(2,1)$ have eigenvectors $(\chi_0, \chi^-, \chi^+), (\tilde{\chi}_0, \tilde{\chi}^-, \tilde{\chi}^+)$. Then they are called *transversal* if $\mathbb{R}^3 = \langle \chi^+, \tilde{\chi}_0, \tilde{\chi}^+ \rangle = \langle \tilde{\chi}^+, \chi_0, \chi^+ \rangle$, where $\langle X_i \rangle$ denotes a subspace generated by X_i. Two elements h_1 and h_2 in $SO(2,1) \ltimes \mathbb{R}^4$ are called *purely hyperbolic* if their linear parts are *purely hyperbolic* and called *transversal* if their linear parts are *transversal*. Note that any purely hyperbolic element and its inverse are transversal.

The orientation on a space like line $l \subset \langle \chi_0, \chi^+ \rangle$ induced by g is defined to be w such that w is parallel to l and $B(w, \chi_0) > 0$. For g_1 and g_2 transversal, $l =$

$\langle \chi_0, \chi^+ \rangle \cap \langle \tilde{\chi}_0, \tilde{\chi}^+ \rangle$ is a spacelike line. Then it is easy to see that the induced orientations on l by g_1 and g_2 are opposite. So if (k,l) and (m,n) are Margulis invariants of two transversal elements $h_1 = (g_1, b)$ and $h_2 = (g_2, c)$ and $km < 0$, then h_1 and h_2 move along the same direction with respect to l in $\mathbb{R}^3$. See [4].

Using the sign of Margulis invariant, Margulis proved in [13] that if $h_1, h_2 \in SO(n+1, n) \ltimes \mathbb{R}^{2n+1}$ are transversal and have opposite signs, then $\langle h_1, h_2 \rangle$ does not act properly on $\mathbb{R}^{n+1,n}$, where $\langle h_i \rangle$ denotes a group generated by h_i.

Proposition 1. *Suppose $h_1 = (A,b), h_2 = (B,c) \in SO(2,1) \ltimes \mathbb{R}^4$ are transversal. Suppose $b = k\chi_0 + le_1 + p\chi^- + q\chi^+$ and $c = m\tilde{\chi}_0 + ne_1 + r\tilde{\chi}^- + s\tilde{\chi}^+$. If Margulis invariants of h_1 and h_2 are $(k,l), (m,n)$, respectively, and $km < 0$, and $\langle k\chi_0 + le_1, \chi^+ \rangle + \frac{p}{1-\lambda^{-1}}\chi^- + \frac{q}{1-\lambda}\chi^+ \cap \langle m\tilde{\chi}_0 + ne_1, \tilde{\chi}^+ \rangle + \frac{r}{1-\tilde{\lambda}^{-1}}\tilde{\chi}^- + \frac{s}{1-\tilde{\lambda}}\tilde{\chi}^+$ is a line L, then the action of the group generated by h_1, h_2 is not proper.*

Proof. Note that, taking inverse if necessary, we may assume that $\ln > 0$. Now $km < 0$ implies that h_1 and h_2 move along the same direction with respect to $\langle \chi^+, \chi_0 \rangle \cap \langle \tilde{\chi}^+, \tilde{\chi}_0 \rangle = L'$. Note that

$$\begin{aligned} L'' = \langle k\chi_0 + le_1, \chi^+ \rangle \cap \langle m\tilde{\chi}_0 + ne_1, \tilde{\chi}^+ \rangle &\subset \langle e_1, \chi_0, \chi^+ \rangle \cap \langle e_1, \tilde{\chi}_0, \tilde{\chi}^+ \rangle \\ &\subset \langle e_1, L' \rangle. \end{aligned}$$

Also $ln > 0, km < 0$ implies that h_1 and h_2 move along the same direction with respect to L''. So h_1 and h_2 move along the same direction with respect to L. Then the proof goes exactly the same way as in the usual Margulis invariant case. Take a rectangle with vertices

$$\frac{p}{1-\lambda^{-1}}\chi^- + \frac{q}{1-\lambda}\chi^+ \pm \frac{1}{2}(k\chi_0 + le_1) \pm \chi^+.$$

The orbit of this rectangle under h_1 will contain L eventually since the linear part of h_1 will dilate χ^+ exponentially. The same thing is true for h_2 with the rectangle

$$\frac{r}{1-\tilde{\lambda}^{-1}}\tilde{\chi}^- + \frac{s}{1-\tilde{\lambda}}\tilde{\chi}^+ \pm \frac{1}{2}(m\tilde{\chi}_0 + ne_1) \pm \tilde{\chi}^+.$$

Choose a compact set K containing these rectangles. Then $h_1^p K \cap h_2^q K \neq \emptyset$ for infinitely many $p, q > 0$. So $h_2^{-q} h_1^p K \cap K \neq \emptyset$. This implies that the action of the group generated by h_1, h_2 is not proper. See [4]. $\square$

The weak point of this theorem is that in general such an intersection line L does not exits. This causes some complications to detect the properness of affine action on $\mathbb{R}^4$ of subgroups in $SO(2,1) \ltimes \mathbb{R}^4$.

3. Margulis invariant determines the affine action

In this section, we show that the affine deformation is completely determined by the Margulis invariant up to translational conjugacy.

Theorem 1. *Suppose $\phi : \Gamma_1 \to \Gamma_2$ is an isomorphism between Zariski dense subgroups consisting of hyperbolic isometries in $SO(2,1) \ltimes \mathbb{R}^4$, which is the identity on the linear part. If ϕ preserves the Margulis invariant, then it is a conjugation by a translation.*

Let $\Gamma \subset O(2,1) \subset O(3,1)$. An *affine deformation* of Γ in $\mathbb{R}^4$ is a homomorphism $\phi : \Gamma \to Iso(\mathbb{R}^{3,1})$ whose linear part is Γ. It is called *proper* if the action on $\mathbb{R}^4$ is proper. Write

$$\phi(g) = (g, u(g)).$$

Then $u = u_\phi$ satisfies the cocycle condition

$$u_\phi(g_1 g_2) = u_\phi(g_1) + g_1 u_\phi(g_2).$$

Such u is called a *cocycle* and the set of cocycles is denoted by $Z^1(\Gamma, \mathbb{R}^4)$. If ϕ_1, ϕ_2 are affine deformations of Γ which are conjugate by translation by v, then the difference $u_{\phi_1} - u_{\phi_2}$ is the cocycle

$$g \to v - gv.$$

Such a cocycle is called a *coboundary*. The set of coboundaries is denoted by $B^1(\Gamma, \mathbb{R}^4)$. Then the set of translational conjugacy classes of affine deformations of Γ is the cohomology $H^1(\Gamma, \mathbb{R}^4) = Z_1(\Gamma, \mathbb{R}^4)/B^1(\Gamma, \mathbb{R}^4)$. Note that $(h,c)^{-1}(g,b)(h,c) = (h^{-1}gh, h^{-1}gc + h^{-1}b - h^{-1}c)$.

If $\chi_0, \chi^\pm$ are eigenvectors of g and

$$b = k\chi_0 + le_1 + \cdots, \quad c = m\chi_0 + ne_1 + \cdots,$$

then

$$h^{-1}gc + h^{-1}b - h^{-1}c = kh^{-1}\chi_0 + le_1 + \cdots \ .$$

Since $h^{-1}gh$ has a positively oriented basis

$$(e_1, h^{-1}\chi_0, h^{-1}\chi^-, h^{-1}\chi^+),$$

the Margulis invariant is invariant under conjugation. The marked Margulis invariant is the function

$$H^1(\Gamma, \mathbb{R}^4) \to (\mathbb{R}^2)^\Gamma,$$

$$[u] \to \alpha_\phi,$$

where $\phi(g) = (g, u(g))$ and $\alpha_\phi(g)$ is the Margulis invariant of $\phi(g)$. The goal of this section is to show that the above function is injective.

A layout of the proof: Let $G = SO(2,1) \ltimes \mathbb{R}^4$. It is not difficult to show that a unique proper maximal normal subgroup of it is $\mathbb{R}^4$. All the normal subgroups of G are $\{e\}, G, \mathbb{R}e_1, 0 + \mathbb{R}^3, \mathbb{R}^4$.

Step I: Suppose two Zariski dense subgroups Γ_1 and Γ_2 are isomorphic by an isomorphism $\rho : \Gamma_1 \to \Gamma_2$. Suppose ρ preserves the Margulis invariant and set $\Gamma = \{(\gamma, \rho(\gamma)) \mid \gamma \in \Gamma_1\}$. Let H be the Zariski closure of Γ in $G \times G$. Denote P_i a projection onto each factor. Then P_i is an isomorphism.

Step II: Then $\phi = P_2 \circ P_1^{-1} : G \to G$ is an isomorphism extending ρ. If the isomorphism is the identity on the linear part, it is a conjugation by a translation.

First we identify $G = SO(2,1) \ltimes \mathbb{R}^4$ with a subgroup of $GL(5, \mathbb{R})$. Any element in G is of the form

$$\begin{bmatrix} 1 & 0 & \xi \\ 0 & SO(2,1) & \\ 0 & 0 & 1 \end{bmatrix},$$

where $\xi \in \mathbb{R}^4$ is a column vector. Let

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then G is an algebraic subgroup of $GL(5, \mathbb{R})$. For G can be defined as

$$\begin{aligned} &\{(x_{ij})_{1 \leqslant i,j \leqslant 5} \mid x_{11} = x_{55} = 1, x_{12} = x_{13} = x_{14} = x_{21} \\ &\quad = x_{31} = x_{41} = x_{51} = x_{52} = x_{53} = x_{54} = 0, M = (x_{ij})_{2 \leqslant i,j \leqslant 4}, MJM^t \\ &\quad = J, \det\ M = 1\}. \end{aligned}$$

We will consider a subset S in $G \times G$. The subset S consists of pairs of elements having the same Margulis invariants up to sign. Since finding eigenvector with eigenvalue 1 and having the same Margulis invariants are polynomial equations, the set S will be an algebraic subvariety of $G \times G$. Another way of putting this is as follows.

Let

$$\mathcal{S} = \left\{ \begin{bmatrix} A & \xi \\ 0 & 1 \end{bmatrix}, X = (0, x_2, x_3, x_4) \in \mathbb{R}^4 \mid A \in SO(2,1), x_2^2 + x_3^2 - x_4^2 = 1, -1, 0, AX = X \right\}.$$

Then $G = SO(2,1) \ltimes \mathbb{R}^4$ can be naturally included in $\mathcal{S}$. See also Section 4. Note that $\mathcal{S}$ is an algebraic subvariety of $GL(5, \mathbb{R}) \times \mathbb{R}^4$. Denote an element in $\mathcal{S}$ by (A, ξ, X). Let

$$\mathcal{T} = \{[(A, \xi, X), (B, \eta, Y)] \in \mathcal{S} \times \mathcal{S} \mid \mathbb{B}(X, \xi) = \mathbb{B}(Y, \eta), \mathbb{B}(\xi, e_1) = \mathbb{B}(\eta, e_1)\}.$$

Note that if $h_1 = (A, b), h_2 = (B, c)$ are hyperbolic, and they have the same Margulis invariant, then naturally $(h_1, h_2) \in \mathcal{T}$.

Then $\mathcal{T} \cap (G \times G)$ is an algebraic subvariety of $G \times G$. So $\mathcal{T} \cap (G \times G)$ is closed in Zariski topology of $G \times G$.

Proposition 2. *Let $\rho : \Gamma_1 \to \Gamma_2$ be an isomorphism between Zariski dense subgroups $\Gamma_1, \Gamma_2 \subset G$ preserving Margulis invariant. Let H be the Zariski closure of $\Gamma = \{(\gamma, \rho(\gamma)) \mid \gamma \in \Gamma_1\}$ in $G \times G$. Denote P_i a projection from H to each factor. Then P_i is an isomorphism.*

Proof. Note $H \subset \mathcal{T} \cap (G \times G)$. Specially this implies that for any $(\alpha, \beta) \in H$,α and β hyperbolic, α and β have the same Margulis invariant up to sign of the first component of the Margulis invariant. First, $P_i(H)$ is normalized by Γ_i, so it is a normal subgroup of G since Γ_i is Zariski dense. But $P_i(H)$ is not contained in $\mathbb{R}^4$ (which is the maximal normal subgroup not equal to $\{e\}, G$), so it must be the whole group G. So P_i is onto.

Suppose P_1 has a kernel K. It is a subgroup of $\{e\} \times G = G$. Since it is normal in H and Γ_2 is Zariski dense in G, K is normal in $G = SO(2,1) \ltimes \mathbb{R}^4$. Since only normal subgroups of G are $\{e\}, G, \mathbb{R}^4, \mathbb{R}, \mathbb{R}^3$, K must be one of these. If $K \neq \{e\}$, there exists $(h_1, h_2) \in H$ whose Margulis invariants are different. More precisely, if $K = \mathbb{R}e_1$. Choose $[h_1 = (A, b), h_2 = (B, c)] \in H$ so that A, B are hyperbolic and the e_1 components of b, c are nonzero (such a pair exists since $P_i H = G$). Since $[(I, 0), (I, \mathbb{R}e_1)] = K$, by multiplying an element in K, one obtains $[h_1, h_2' = (B, d)] \in H$ with the e_1 component of d equal zero. Then the Margulis invariants of h_1, h_2' are different. But since $H \subset \mathcal{T}$, Margulis invariants of h_1 and h_2' should be the same. This is a contradiction. If $K = \{0\} + \mathbb{R}^3$, take $[h_1 = (A, b), h_2 = (B, c)]$ so that $\mathbb{B}(b, \chi_0(A)) = \mathbb{B}(c, \chi_0(B)) \neq 0$. But multiplying an element in K, one can get $[h_1, h_2' = (B, d)] \in H$ so that $\mathbb{B}(\chi_0(B), c) \neq \mathbb{B}(\chi_0(B), d)$. This is again a contradiction. The same thing is true for $K = G$ or $K = \mathbb{R}^4$. This shows that K is trivial and so P_1 is an isomorphism. Similarly P_2 is an isomorphism. $\square$

Now take $\phi = P_2 \circ P_1^{-1} : G \to G$ which is an isomorphism extending $\rho : \Gamma_1 \to \Gamma_2$. This is a continuous map since projection is a continuous map. Note that ϕ still preserve the Margulis invariant up to sign for hyperbolic elements.

Now suppose Γ_i comes from an affine deformation of a Zariski dense subgroup of $SO(2,1)$. Then ϕ will preserve the linear part by the definition of an affine deformation. So $\phi(A, b) = (A, c)$ for any $(A, b) \in G$. We want to show that ϕ is a conjugation by a translation. First we prove several propositions.

Proposition 3. *Let $\phi : G \to G$ be a Margulis invariant preserving continuous isomorphism which is the identity on linear part. Then $\phi(\mathbb{R}^4) = \mathbb{R}^4$ and indeed it is the identity on $\mathbb{R}^4$.*

Proof. Since $\mathbb{R}^4$ is the maximal normal subgroup of $G = SO(2,1) \rtimes \mathbb{R}^4$, and since ϕ is an isomorphism, $\phi(\mathbb{R}^4)$ must be a maximal normal subgroup of G again, which is $\mathbb{R}^4$. Since the group structure of $I \times \mathbb{R}^4$ is just the addition in $\mathbb{R}^4$, $\phi|_{\mathbb{R}^4}$ must be a linear map. Denote it by L. We want to show that L is the identity.

Let $\phi(A,0) = (A, u_A)$ and fix a hyperbolic element $A \in SO(2,1)$. Denote the eigenvectors of A by $\chi_A^0, \chi_A^-, \chi_A^+$ as usual and let $u_A = k'\chi_A^0 + l'e_1 + m'\chi_A^- + n'\chi_A^+$. For $b \in \mathbb{R}^4$, let $b = k\chi_A^0 + le_1 + m\chi_A^- + n\chi_A^+$. Note $\phi(A,b) = \phi(A,0)\phi(I, A^{-1}b) = (A, ALA^{-1}b + u_A)$.

Set

$$ALA^{-1}\chi_A^0 = x_{11}\chi_A^0 + x_{12}e_1 + x_{13}\chi_A^- + x_{14}\chi_A^+,$$

$$ALA^{-1}e_1 = x_{21}\chi_A^0 + x_{22}e_1 + x_{23}\chi_A^- + x_{24}\chi_A^+,$$

$$ALA^{-1}\chi_A^- = x_{31}\chi_A^0 + x_{32}e_1 + x_{33}\chi_A^- + x_{34}\chi_A^+,$$

$$ALA^{-1}\chi_A^+ = x_{41}\chi_A^0 + x_{42}e_1 + x_{43}\chi_A^- + x_{44}\chi_A^+.$$

Since ϕ preserves the Margulis invariant

$$kx_{11} + lx_{21} + mx_{31} + nx_{41} + k' = k,$$

$$kx_{12} + lx_{22} + mx_{32} + nx_{42} + l' = l.$$

But this must be true for any choice of b, so for any choice of k, l, m, n. So we get

$$x_{11} = x_{22} = 1, \quad k' = l' = x_{21} = x_{31} = x_{41} = x_{12} = x_{32} = x_{42} = 0.$$

This implies that

$$ALA^{-1}\chi_A^0 = \chi_A^0 + x_{13}\chi_A^- + x_{14}\chi_A^+, \tag{1}$$

$$ALA^{-1}e_1 = e_1 + x_{23}\chi_A^- + x_{24}\chi_A^+, \tag{2}$$

$$ALA^{-1}\chi_A^- = x_{33}\chi_A^- + x_{34}\chi_A^+, \tag{3}$$

$$ALA^{-1}\chi_A^+ = x_{43}\chi_A^- + x_{44}\chi_A^+. \tag{4}$$

Then Eqs. (3) and (4) imply that L leaves invariant $\langle \chi_A^-, \chi_A^+ \rangle$ for any hyperbolic element $A \in SO(2,1)$.

If $L\chi_A^+$ is a scalar multiple of χ_A^+ for all hyperbolic A, L restricted to $0 + \mathbb{R}^3$ must be a scalar multiple of the identity in $\mathbb{R}^3$, say μI. Then

$$ALA^{-1}\chi_A^0 = \mu\chi_A^0.$$

Since $x_{11} = 1$, we conclude that L restricted to $0 + \mathbb{R}^3$ is the identity.

So suppose there is a hyperbolic element A so that $L\chi_A^+ \notin \langle \chi_A^+ \rangle$. Choose a hyperbolic element $B \in SO(2,1)$ so that $\chi_A^+ = \chi_B^+, \chi_B^- \neq \chi_A^-$ and $L\chi_A^+ \notin \langle \chi_B^-, \chi_B^+ \rangle$. This is not allowed since $L\chi_A^+ = L\chi_B^+$ must be in $\langle \chi_B^-, \chi_B^+ \rangle$. This way one can shows that $L\chi_A^+ = \chi_A^+$ for any hyperbolic A. Then it is easy to see that by varying A, L must be identity on $0+\mathbb{R}^3$. Eq. (2) implies that

$$ALA^{-1}e_1 = ALe_1 = e_1 + x_{23}\chi_A^- + x_{24}\chi_A^+$$

for any hyperbolic $A \in SO(2,1)$. So

$$Le_1 = e_1 + x_{23}\lambda_A^{-1}\chi_A^- + x_{24}\lambda_A\chi_A^+$$

for any A, which implies that $Le_1 = e_1$. So $L = I$ on $\mathbb{R}^4$. $\square$

Now we finish the proof.

Proposition 4. *Let ϕ be as in the previous proposition. Then ϕ is a conjugation by a vector C.*

Proof. Take a canonical Cartan (polar) decomposition $SO(2,1) = K\bar{A}^+K$ where $\bar{A}^+$ is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{bmatrix}$$

and K is of the form

$$\begin{bmatrix} SO(2) & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$A_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{bmatrix}.$$

Note $\chi^0_{A_t} = (-1,0,0), \chi^-_{A_t} = (0,-1,1), \chi^+_{A_t} = (0,1,1)$ for $t>0$. If $\phi(A_t) = (A_t, u)$, then u is in $\langle \chi^-_{A_t}, \chi^+_{A_t} \rangle$ because ϕ preserves Margulis invariant. There exists a unique solution in $\langle \chi^-_{A_t}, \chi^+_{A_t} \rangle$ to $A_tC_t - C_t = u$. So from now on we write $\phi(A_t) = (A_t, A_tC_t - C_t)$. Further we identify A_t with

$$\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

Note $\phi(A_t)$ has a fixed point $-C_t$. But it has a unique fixed point since $A_tX + AC_t - C_t = X$ has a unique solution $X = -C_t$. Since A_t is abelian, $\{A_t\}$ has a unique global fixed point $-C_t = -C$. This shows that

$$\phi(A_t) = (A_t, A_tC - C).$$

So far we showed that $\phi(A) = (A, AC - C)$ for any $A \in A^+$.

Now we do the same thing for K. Since ϕ is continuous, $\phi(K)$ is compact. Since $\phi(A) = (A, u_A) \in K \ltimes \mathbb{R}^4$, $\phi(K)$ is a Euclidean isometry subgroup of $Iso(\mathbb{R}^4)$. So it has a global fixed point $-C'$. This implies that $-AC' + u_A = -C'$. So $\phi(A) = (A, AC' - C')$ for any $A \in K$. Let $C' = (0, b, a, c)$. Let $C = (0, 0, x, y)$. Since $A \in K$ is of the form

$$\begin{bmatrix} SO(2) & 0 \\ 0 & 1 \end{bmatrix},$$

one can take $C' = (0, b, a, y)$. Similarly since A^+ fixes $(1, 0, 0)$, one can take $C = (0, b, x, y)$. So all we have to show is $x = a$.

Note that there exists $k \in K$ so that $kA_t k = A_{-t}$. Explicitly

$$k = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\phi(kA_t k) = \phi(A_{-t})$ implies that

$$(k, kC' - C')(A_t, A_t C - C)(k, kC' - C') = (A_t^{-1}, A_t^{-1} C - C).$$

Then a direct calculation shows that

$$A_{-t}(kC - kC' + C' - C) = kC - kC' + C' - C.$$

This is possible only when $a = x$ since A^+ fixes $\mathbb{R}(1, 0, 0)$.

Since $\phi(A) = (A, AC - C)$ for $A \in A^+$ and $A \in K$, using $SO(2, 1) = K\bar{A}^+ K$, a direct calculation shows that $\phi(A) = (A, AC - C)$ for any $A \in SO(2, 1)$. Since $\phi|_{\mathbb{R}^4} = I$, $\phi(A, b) = (A, b + AC - C)$. So ϕ is a conjugation by C. □

Since we identified $SO(2, 1)$ with

$$\begin{bmatrix} 1 & 0 \\ 0 & SO(2,1) \end{bmatrix}$$

the same proof works for $SO(2, 1) \ltimes \mathbb{R}^3$. So we have

Corollary 1. *Suppose two isomorphic Zariski dense subgroups Γ_1 and Γ_2 consisting of hyperbolic elements with the identical linear parts in $SO(2, 1) \ltimes \mathbb{R}^3$, have the same Margulis invariant. Then they are conjugate by a translation.*

This corollary is independently proved by [5]. In the next section, we prove this fact for all Zariski dense subgroups in $SO(n, n-1) \ltimes \mathbb{R}^{2n-1}$.

4. Isospectral rigidity of $SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$

Before we proceed with a proof, it is best to give an example. Take $SO(3,2)$. Let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cosh t & 0 & 0 & \sinh t \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \sinh t & 0 & 0 & \cosh t \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cosh t & \sinh t & 0 \\ 0 & 0 & \sinh t & \cosh t & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Set $\chi_0 = e_1 = (1, 0, \dots, 0), \chi_{\pm 1} = (0, \pm 1, 0, 0, 1),\ \chi_{\pm 2} = (0, 0, \pm 1, 1, 0)$. Let

$$B_1 = \mathrm{Diag}(-1, -1, 1, 1, 1), \quad B_1' = \mathrm{Diag}(-1, 1, 1, 1, , -1),$$

$$B_2 = \mathrm{Diag}(-1, 1, -1, 1, 1), \quad B_2' = \mathrm{Diag}(-1, 1, 1, -1, 1)$$

be elliptic elements so that

$$B_1 A_1 B_1 = B_1' A_1 B_1' = A_1^{-1}, \quad B_2 A_2 B_2 = B_2' A_2 B_2' = A_2^{-1},$$

$$B_i A_j B_i = A_j, \quad B_i' A_j B_i' = A_j, \quad i \neq j.$$

If

$$\begin{aligned} C &= x_1 \chi_{-1} + y_1 \chi_1 + x_2 \chi_{-2} + y_2 \chi_2 \\ &= (0, -x_1 + y_1, -x_2 + y_2, x_2 + y_2, x_1 + y_1), \end{aligned}$$

then

$$B_1 C - C = (0, 2(x_1 - y_1), 0, 0, 0), \quad B_1' C - C = (0, 0, 0, 0, -2(x_1 + y_1))$$

$$B_2 C - C = (0, 0, 2(x_2 - y_2), 0, 0), \quad B_2' C - C = (0, 0, 0, -2(x_2 + y_2), 0).$$

So if

$$\begin{aligned} C' &= z_1 \chi_{-1} + w_1 \chi_1 + z_2 \chi_{-2} + w_2 \chi_2 \\ &= (0, -z_1 + w_1, -z_2 + w_2, z_2 + w_2, z_1 + w_1) \end{aligned}$$

and

$$B_i C - C = B_i C' - C', \quad B_i' C - C = B_i' C' - C',$$

then $C = C'$. We will use these in the next theorem. One should keep in mind the above example.

Fix a Cartan decomposition $SO(n+1,n) = K\bar{A}^+K$. Since the symmetric space $SO(n+1,n)/SO(n+1) \times SO(n)$ has rank n, and since one can embed $SO(2,1)/SO(2)$ into it, there exists $A_1, \dots, A_n \in \bar{A}^+$ and $\chi_0 = \pm e_1, \{\chi_{-i}, \chi_i\}_{i=1,\dots,n}$ so that A_i generate A^+ and

$$A_i\chi_{-i} = \lambda_{-i}\chi_{-i}, A_i\chi_i = \lambda_i\chi_i.$$

For $i \neq j$,

$$A_i\chi_{-j} = \chi_{-j}, A_i\chi_j = \chi_j.$$

Every A_i fixes χ_0. For each A_i, there exists $B_i = B_i^{-1},\ B_i' = B_i'{-1} \in K$ so that

$$B_iA_iB_i = A_i^{-1}, \quad B_i'A_iB_i' = A_i^{-1}.$$

Specially B_i swaps χ_{-i} and χ_i, and fixes all χ_k, χ_{-k} for $i \neq k$. Similarly

$$B_i'\chi_i = -\chi_{-i}, B_i'\chi_{-i} = -\chi_i$$

and fixes all χ_k, χ_{-k} for $i \neq k$.

Note that for $A \in A^+$, the Margulis invariant of (A, b) with $b = m\chi_0 + \sum l_i\chi_i$, is m. Also the eigenvalues of any element A in the interior A^+ of the closed Weyl chamber $\bar{A}^+$, are all different from 1 except the one corresponding to $\chi_0 = \pm e_1$.

Set

$$\mathcal{S} = \left\{ \begin{bmatrix} A & \xi \\ 0 & 1 \end{bmatrix}, X \in \mathbb{R}^{2n+1} \mid A \in SO(n+1,n), AX = X, \mathbb{B}(X,X) = -1, 1, 0 \right\}.$$

Then $G = SO(n+1,n) \ltimes \mathbb{R}^{2n+1}$ can be naturally included in $\mathcal{S}$ in the following sense. If (A,ξ) is hyperbolic, X corresponds to an eigenvector with eigenvalue 1, of course, it is not unique since we can equally take $-X$. But it is clear in the proof of Proposition 2, that the sign is not important (i.e., the same absolute value of Margulis invariant is sufficient). If (A,ξ) is not hyperbolic, there might be a lot of X corresponding to this element, for example if one takes (I,ξ), X is a set in $\mathbb{R}^{2n+1}$ with norm ± 1. But again it is clear in Proposition 2 that all we use are hyperbolic elements.

Note that $\mathcal{S}$ is an algebraic subvariety of $GL(2n+2,\mathbb{R}) \ltimes \mathbb{R}^{2n+1}$. Denote an element in $\mathcal{S}$ by (A,ξ,X). Let

$$\mathcal{T} = \{[(A,\xi,X),(B,\eta,Y)] \in \mathcal{S} \times \mathcal{S} \mid \mathbb{B}(\xi,X) = \mathbb{B}(\eta,Y)\}.$$

Note that if $h_1 = (A,b), h_2 = (B,c)$ are hyperbolic, and have the same Margulis invariant, then naturally $(h_1,h_2) \in \mathcal{T}$. Then the same argument works for $SO(n+1,n)$ to prove Proposition 2.

We show Propositions 3 and 4 in this case.

Proposition 5. *Let $\phi : G \to G$ be a Margulis invariant preserving continuous isomorphism, which is the identity on linear part. Then $\phi|_{\mathbb{R}^{2n+1}} = I$.*

Proof. Denote $\phi|_{\mathbb{R}^{2n+1}} = L$. For a purely hyperbolic element A, denote χ_i^A, $-n \leqslant i \leqslant n$ its eigenvectors as usual. If $\phi(A,0) = (A, u_A)$, $\phi(A,b) = (A, ALA^{-1}b + u_A)$ for any $b = \sum y_i \chi_i^A$. Set $u_A = \sum y_i' \chi_i^A$, $-n \leqslant i \leqslant n$. Set

$$ALA^{-1}\chi_i^A = \sum x_{ij}\chi_j^A, \quad -n \leqslant i, j \leqslant n.$$

Then

$$ALA^{-1}b = \sum y_k x_{kj} \chi_j^A, \quad -n \leqslant k, \quad j \leqslant n.$$

Since (A,b) and $\phi(A,b)$ have the same Margulis invariant,

$$y_0 = \sum_k y_k x_{k0} + y_0'.$$

But y_k is arbitrary, so

$$x_{00} = 1, \quad y_0' = 0, \quad x_{k0} = 0, \quad k \neq 0.$$

This implies that for $i \neq 0$,

$$ALA^{-1}\chi_i^A \in \langle \chi_j^A \rangle_{j \neq 0}$$

for any purely hyperbolic element A. So

$$L\langle \chi_j^A \rangle_{j \neq 0} \subset \langle \chi_j^A \rangle_{j \neq 0}.$$

As in Proposition 3, if $L\chi_1^A$ is a scalar multiple of χ_1^A for all hyperbolic elements A, then L is μI. But since $x_{00} = 1$, $\mu = 1$.

So suppose for some hyperbolic element A, $L\chi_1^A$ is not a scalar multiple of χ_1^A. As in Proposition 3, choose a hyperbolic element B so that $\chi_1^B = \chi_1^A$ and $L\chi_1^B \notin \langle \chi_j^B \rangle_{j \neq 0}$, which will give a contradiction. So $L = I$. $\square$

Now we finish the proof in general case.

Theorem 2. *Let $\Gamma_1, \Gamma_2 \subset G = SO(n+1, n) \ltimes \mathbb{R}^{2n+1}$ be isomorphic Zariski dense subgroups consisting of hyperbolic elements with the identical linear parts. If they have the same Margulis invariant, then they are conjugate by a translation.*

Proof. As before let $\phi : G \to G$ be the Margulis invariant preserving continuous isomorphism extending a given isomorphism between Γ_1 and Γ_2. For any $A \in A^+$, since ϕ preserves the Margulis invariant, if $\phi(A) = (A, u_A)$, then u_A is contained in $\langle \chi_{-i}, \chi_i \rangle_{i \neq 0}$. So $AC_A - C_A = u_A$ has a unique solution in $\langle \chi_{-i}, \chi_i \rangle_{i \neq 0}$ since A has eigenvalues all different from 1 except the one corresponding to χ_0. Also $AX + AC_A - C_A = X$ has a unique solution $X = -C_A$ in $\langle \chi_{-i}, \chi_i \rangle_{i \neq 0}$. Since A^+ is abelian, $\phi(A^+)$ must fix a unique fixed point in $\langle \chi_{-i}, \chi_i \rangle_{i \neq 0}$, so $C_A = C$ is universal for any $A \in A^+$. So for any A in A^+,

$$\phi(A) = (A, AC - C).$$

Since $\phi(K) \subset SO(n+1) \times SO(n) \ltimes \mathbb{R}^{2n+1}$ is a compact group in Euclidean isometry group, it has a global fixed point $-C'$ in $\mathbb{R}^{2n+1}$. So if $\phi(A) = (A, u_A)$, then $-AC' + u_A = -C'$ for any $A \in K$ and so $\phi(A) = (A, AC' - C')$.

Now we show that C can be chosen as C'. Set

$$C = \sum_{k \neq 0} x_k \chi_{-k} + \sum_{k \neq 0} y_k \chi_k, \; C' = te_1 + \sum_{k \neq 0} z_k \chi_{-k} + \sum_{k \neq 0} w_k \chi_k.$$

Using $B_i A_i B_i = A_i^{-1}$, $\phi(B_i A_i B_i) = \phi(A_i^{-1})$ implies that

$$A_i^{-1}(B_i C - B_i C' + C' - C) = B_i C - B_i C' + C' - C.$$

Since A_i fixes only $\mathbb{R}e_1$, we obtain $x_i - y_i = z_i - w_i$. Similarly using $B_i' A_i B_i' = A_i^{-1}$, we obtain $x_i + y_i = z_i + w_i$. This shows that $C + te_1 = C'$. But since $Ae_1 = e_1$ for all $A \in A^+$, $AC' - C' = AC - C$ for all A in A^+. This shows that C can be taken equal to C'. Let $C = C'$. Then $\phi(A) = (A, AC - C)$ for any $A \in A^+$ or $A \in K$. Since $\phi|_{\mathbb{R}^{2n+1}} = Id$, and using Cartan decomposition $SO(n+1, n) = K\bar{A}^+K$,

$$\phi(A, b) = (A, AC - C + b).$$

So ϕ is a conjugation by C. $\square$

In fact, the following theorem is true.

Theorem 3. *Let $\phi : \Gamma_1 \to \Gamma_2$ be an isomorphism between two Zariski dense subgroups of $SO(n+1, n) \ltimes \mathbb{R}^{2n+1}$ preserving the Margulis invariant. Then Γ_1 and Γ_2 are conjugate.*

Proof. As before we obtain an extention of ϕ, denoted again by ϕ, $\phi : G \to G$ which preserves the Margulis invariant. We do not have a property that $\phi(A, b) = (A, c)$ anymore, so we directly work on the linear group $SO(n+1, n)$ which is semisimple. We claim that ϕ induces an isomorphism on $SO(n+1, n)$.

To define a map on $SO(n+1, n)$ induced from ϕ, we have to show that, for any $A \in SO(n+1, n)$, $\phi(A, \mathbb{R}^{2n+1}) = (\Pi(\phi(A, 0)), \mathbb{R}^{2n+1})$ where $\Pi : G \to SO(n+1, n)$ is a projection onto $SO(n+1, n)$.

As before, since $I \times \mathbb{R}^{2n+1}$ is a unique maximal normal subgroup of G and since ϕ is an isomorphism, $\phi(I \times \mathbb{R}^{2n+1}) = I \times \mathbb{R}^{2n+1}$.

Suppose $\phi(A,b)=(B,c), \phi(A,d)=(D,e)$ for $A\in\Pi(\Gamma_1), B\neq D\in\Pi(\Gamma_2)$ hyperbolic elements. Then

$$\phi((A,b)(A,d)^{-1})=\phi(I,b-d)=(BD^{-1},c-BD^{-1}e).$$

This contradicts the fact that $\phi(I\times\mathbb{R}^{2n+1})=I\times\mathbb{R}^{2n+1}$.

So there exists a well-defined map $\bar{\phi}: SO(n+1,n)\to SO(n+1,n)$ so that

$$\bar{\phi}\circ\Pi=\Pi\circ\phi.$$

Certainly by the definition of $\bar{\phi}$, $\bar{\phi}$ is a continuous isomorphism of $SO(n+1,n)$. Then by the theory of semisimple Lie groups, see [6, p. 247], $\bar{\phi}$ is a conjugation by some element $\alpha\in SO(n+1,n)$.

After conjugation of ϕ by α, we obtain an isomorphism $\rho: G\to G$ so that $\rho(A,b)=(A,c)$, i.e., ρ is the identity on the linear part. Now we apply Theorem 2 to conclude that Γ_1 and Γ_2 are conjugate. $\square$

5. Affine deformation of $SO(2,1)$ in $\mathbb{R}^4$

In this section we closely follow [7]. As in Section 2, we embed $SO(2,1)$ into $SO(3,1)$ by

$$\begin{bmatrix} 1 & 0 \\ 0 & SO(2,1) \end{bmatrix}.$$

In this section, we concern a quasifuchsian deformation of a purely hyperbolic subgroup $\Gamma\in SO(2,1)$ in $SO(3,1)$. If $\phi(g)=(g,u_g)$ is an affine deformation of Γ in $\mathbb{R}^4$, u is an element in $Z^1(\Gamma,\mathbb{R}^4)$ as explained in Section 3. By considering $\mathbb{R}^4$ as a subspace of $\mathbb{R}^6=\mathfrak{so}(3,1)$, one can view u as an element in $Z^1(\Gamma,\mathfrak{so}(3,1))$.

Lemma 2. *There exits a map $\psi:\mathbb{R}^3\to\mathfrak{so}(2,1)$ which is Ad-equivariant. Explicitly ψ is*

$$(y,z,w)\to\begin{bmatrix} 0 & -w & z \\ w & 0 & y \\ z & y & 0 \end{bmatrix}.$$

Proof. Using $SO(2,1)=K\bar{A}^{+}K$ where K is of the form

$$\begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and A^+ is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{bmatrix},$$

it suffices to show that ψ is the right map for each g in K or A^+. Suppose $g \in A^+$. Then

$$g(y,z,w) = (y, z\cosh t + w\sinh t, z\sinh t + w\cosh t).$$

On the other hand,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{bmatrix} \begin{bmatrix} 0 & -w & z \\ w & 0 & y \\ z & y & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh t & -\sinh t \\ 0 & -\sinh t & \cosh t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -w\cosh t - z\sinh t & w\sinh t + z\cosh t \\ w\cosh t + z\sinh t & 0 & y \\ w\sinh t + z\cosh t & y & 0 \end{bmatrix}.$$

This shows that

$$g(y,z,w) = g\psi(y,z,w)g^{-1}.$$

A similar calculation holds for K. $\square$

Similarly there exits an Ad-equivariant map $\psi : \mathbb{R}^6 \to \mathfrak{so}(3,1)$. By the choice of our embedding of $SO(2,1)$ in $SO(3,1)$, it is obvious that ψ map (x,y,z,w) to

$$\begin{bmatrix} 0 & * & * & * \\ * & 0 & -w & z \\ * & w & 0 & y \\ * & z & y & 0 \end{bmatrix}.$$

Let $h = (g,b) \in SO(2,1) \ltimes \mathbb{R}^4$ be a hyperbolic element such that $b = (x,y,z,w)$ and

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh t & \sinh t \\ 0 & 0 & \sinh t & \cosh t \end{bmatrix}.$$

Note $\chi_g^0 = (0, -1, 0, 0)$, and so the Margulis invariant of h is

$$(\chi_g^0 \text{ component}, x) = (-y, x).$$

Then viewing b as an element in $\mathfrak{so}(3,1)$,

$$tr(bg) = tr(\psi(b)g) = 2y \sinh t.$$

Also

$$\sqrt{(tr(g) - 2)^2 - 4} = 2 \sinh t.$$

So

$$\chi_g^0 \text{ component} = -\frac{tr(bg)}{\sqrt{(tr(g) - 2)^2 - 4}}.$$

In general if $h = (A, b) \in SO(2,1) \ltimes \mathbb{R}^4$ is hyperbolic, there exits $B \in SO(2,1)$ so that

$$BAB^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh t & \sinh t \\ 0 & 0 & \sinh t & \cosh t \end{bmatrix}.$$

Then $B(A, b)B^{-1} = (BAB^{-1}, Bb)$. Since the Margulis invariant is conjugate invariant,

$$\begin{aligned} \chi_A^0 \text{ component} &= -\frac{tr(Bb \cdot BAB^{-1})}{\sqrt{(tr(BAB^{-1}) - 2)^2 - 4}} \\ &= -\frac{tr(bA)}{\sqrt{(trA - 2)^2 - 4}}. \end{aligned}$$

The second equality holds since $\psi(Bb) = B\psi(b)B^{-1}$. So the Margulis invariant of any hyperbolic element $h = (g, b) \in SO(2,1) \ltimes \mathbb{R}^4$ is

$$\left(-\frac{tr(bg)}{\sqrt{(tr(g) - 2)^2 - 4}}, e_1 \text{ component of } b \right) \tag{5}$$

where b is identified to an element in $\mathfrak{so}(3,1)$ under ψ.

An element $u \in Z^1(\Gamma, \mathbb{R}^6)$, satisfies

$$u(g_1 g_2) = u(g_1) + g_1 u(g_2).$$

Identifying $\mathbb{R}^6$ with $\mathfrak{so}(3,1)$ via ψ,

$$\psi(u(g_1g_2)) = \psi(u(g_1)) + g_1\psi(u(g_2))g_1^{-1},$$

i.e., $u \in Z^1(\Gamma, \mathfrak{so}(3,1)_{Ad})$. So there is a smooth path λ_t in $\mathrm{Hom}(\Gamma, SO(3,1))$ so that $\frac{d}{dt}|_{t=0}\lambda_t = u,\ \lambda_0 = \Gamma$ and

$$\lambda_t(g) = g\exp(tu(g) + O(t^2)). \tag{6}$$

If α is an isometry of a Riemannian manifold X, the translation length $l(\alpha)$ is defined as

$$\min_{x\in X} d(x, \alpha x).$$

Proposition 6. *Let $\phi : \Gamma \to SO(2,1) \ltimes \mathbb{R}^4$ be an affine deformation of a discrete subgroup $\Gamma \subset SO(2,1)$ with a cocycle $u \in Z^1(\Gamma, \mathbb{R}^4) \subset Z^1(\Gamma, \mathfrak{so}(3,1))$. Suppose that $\mu(t)$ is a path in Hom$(\Gamma, SO(3,1))$ so that $\mu(0) = \Gamma, \mu'(0) = u$. Then the derivative $L_g'(0)$ at* 0 *of a translation length $l(\mu_t(g))$ is*

$$L_g'(0) = -\frac{tr(u(g)g)}{\sqrt{(tr(g)-2)^2 - 4}}.$$

Proof. Note $\mu(t) : \Gamma \to SO(3,1)$ is a deformation of a Fuchsian group Γ. Since the set of quasifuchsian deformations of a closed hyperbolic surface is open in $\mathrm{Hom}(\Gamma, SO(3,1))$, see [16], $\mu(t)$ is a quasifuchsian deformation of Γ for small t. Also there exists a smooth curve tangent to u so that

$$\lambda_t(g) = g\exp(tu(g) + O(t^2))$$

for $g \in \Gamma$ by Eq. (6).

Then

$$\begin{aligned}\frac{d}{dt}|_{t=0} tr\ \mu_t(g) &= \frac{d}{dt}|_{t=0} tr\ \lambda_t(g) = \frac{d}{dt}|_{t=0} tr(g\exp(tu(g) + O(t^2))) \\ &= \frac{d}{dt}|_{t=0} tr(g(I + tu(g) + O(t^2))) = tr(gu(g)).\end{aligned}$$

By Eq. (5),

$$\frac{d}{dt}|_{t=0} tr(\mu_t(g)) = -\sqrt{(tr(g)-2)^2 - 4}\cdot \chi_g^0 \text{ component.}$$

Set $L_g(t) = l(\mu_t(g))$. Since $tr\mu_t(g) = 2 + 2\cosh\left(\frac{l(\mu_t(g))}{2}\right)$, we get

$$\frac{d}{dt}|_{t=0} tr\mu_t(g) = \sinh\left(\frac{l(g)}{2}\right)L_g'(0).$$

Since $\sqrt{(tr(g)-2)^2-4} = 2\sinh\frac{l(g)}{2}$, we get

$$\chi_g^0 \text{ component} = -\frac{L_g'(0)}{2}. \qquad \square$$

If $\phi : \Gamma \to SO(2,1) \ltimes \mathbb{R}^4$ is an affine deformation with $u = u_\phi \in Z^1(\Gamma, \mathbb{R}^4)$ such that

$$-\frac{tr(u(g)g)}{\sqrt{(tr(g)-2)^2-4}} > 0$$

for all $g \in \Gamma$, we call u_ϕ *positive*.

Now we are ready to state the following

Theorem 4. *Suppose $\Gamma \subset SO(2,1)$ is a cocompact lattice. Then any affine deformation $\phi : \Gamma \to SO(2,1) \ltimes \mathbb{R}^4$ with u_ϕ positive is not proper.*

Proof. By Proposition 6, we have that $L_g'(0) > 0$ for all $g \in \Gamma$. This implies that for any $g \in \Gamma$, $l(\mu_t(g))$ is increasing, i.e., the quasifuchsian group μ_t has a larger translation length $l(\mu_t(g))$ than $l(g)$. Let S be a hyperbolic surface H^2/Γ. One can construct a hyperbolic pleated surface S_t in $M_t = H^3/\mu_t$ as in [17]. S_t is a complete hyperbolic surface with respect to the path metric on S_t. Obviously for any $\gamma \in \pi_1(M_t)$, $l([\gamma])$ in M_t is smaller than the corresponding one in S_t. This implies that for any $\gamma \in \pi_1(S)$, $l([\gamma])$ in S_t is greater than the corresponding one in S. So all the closed geodesics lengthen in S_t, which is impossible for closed hyperbolic surface. See also the argument in [7]. $\square$

Originally the above theorem in $\mathbb{R}^3$ is due to Mess [14] and recently it is reproved by [7].

Open question: When one uses an irreducible representation of $SO(2,1)$ in $\mathbb{R}^n$, Labourie proved for any dimension that any cocompact lattice in $SO(2,1)$ does not have a proper affine deformation [10]. We wonder whether we can drop the hypothesis that u is positive (or negative) even in this reducible representation case.

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